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FOR THE COMMANDER

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L.G. TAFF

Group 94

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ABSTRACT

This Report deals with a number of interrelated topics. The common thread is Laplace's method of initial orbit determination based on passively acquired optical data. We discuss this method's principal competitor (that of Gauss), the difficulties of Gauss's technique, and the traditional reasons the Gaussian method is preferred to the Laplacian. We reject this hegemony for a variety of reasons and concentrate on Laplace's method in an era of a surfeit of high quality data. This leads us into a discussion of data smoothing. Once one leaves the raw observatorial data the possibility of combining observations from multiple observers comes to mind and hence the determination of parallax by trigonometrical means. All of this may be applied to two different classes of objects - asteroids and artificial satellites. Our immediate interests are in fast moving asteroids ($>0\overset{\circ}{.}5$/day or an abnormally fast ecliptic latitude rate) and high altitude artificial satellites ($P > 6^h$). In both instances it is the high inclination and high eccentricity subset which is of especial concern.

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I. OVERVIEW

This Report is about initial orbit determination utilizing passively acquired angles only data - a classical problem in astronomy. So well developed is the literature on the subject the educated reader might well wonder what might be added. My rejoinder is the following question; "How often have you really computed an orbit given only three sets of optical observations when it was of importance to do so?", and "Assuming that your answer was yes, did it work?". I would hazard to guess that less than 1/10 of one per cent of the readers of this report can answer yes to the first question. Further, I will speculate that the answer to the second question has always been "No". Moreover with the advent of ballistic missiles and artificial satellites and the contemporaneous development and refinement of radars, many of the classical astronomical techniques were suitably refined for these new problems with their new observables. The use of laser radars, beacon tracking, electro-optical cameras, etc. have increased the meaningful data acquisition rate on near-Earth objects both because of their mode of operation and because near-Earth objects move so much faster than do the natural objects within the solar system. As an example a typical minor planet has a geocentric angular speed of $0\overset{\circ}{.}25$/day, a typical high-altitude satellite has a topocentric angular speed of 15"/sec (=360°/day) while a low altitude satellite might have a topocentric angular speed in excess of 10°/min (=14,400°/day). And yet only one initial orbit determination technique breaks with those of the past (see references 1 and 2).

Allow me to elaborate further.

A. Astronomical Initial Orbit Determination

Consider the instances in astronomy when one computes orbits. We might do it for a binary star system or for a moon of another planet. Obviously, in these two cases, there is no danger of losing the object due to an inaccurate set of orbital elements. Nor are there any impediments to the acquisition of an arbitrarily large amount of data.* Thus, except for curiosity or the challenge of being able to deduce an orbit from minimal information, there is no compelling observational need to use only three sets of observations nor is there any compelling need to compute an accurate orbit with alacrity for these types of objects. Not that I neglect the power of curiosity nor the drive of ego. The latter has dominated the entire subject and the former goads all scientists on.

There are three other instances in astronomy where initial orbit determination is practiced - planets, comets, and meteors. There could be a problem for these objects before the advent of photography, but not after. The reasons are several -- photographic plates cover large areas of the sky obviating the need for precision pointing; photographic plates yield a permanent record and old plates can always be searched for pre-discovery images thereby yielding more than the minimum number of observations; and photographic plates were developed at about the same time as rapid world wide communication so that local events lost their ability to adversely influence the making of an observation. Now to the pre-photographic era.

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* Yes planets go into conjunction and most stars set between observing seasons. The planets always return to opposition and the stars always rise again.

Meteors are so common, the sporadic rate is $\sim$10/hour, that one never contemplated computing orbits for any but a few. Since a parallax was determined for a meteor only in 1798, and scientific interest in the subject wasn't stimulated until the great Leonid shower of 1833, I'll take the position that they have played a very minor role in celestial mechanics.

Comets are another case in two regards. Most of them that are naked eye visible are not periodic (Halley's is a notable exception). Hence, recovery was never in question. Moreover, comets have comae and tails and are difficult to lose or lose sight of. In addition Olbers (in 1797) solved the initial orbit problem for parabolic comets and this is but a sidelight on the elliptical problem.

This leaves us planets. Planets were important (at least the modern major ones and the first few minor ones). Remember though that Uranus (discovered by Herschel in 1781) was naked-eye visible and 19 (some sources indicate 20) pre-discovery observations were quickly found. Also with a sidereal period of 83.7 years one was not in much danger of losing Uranus. As both Neptune and Pluto were discovered on the basis of predictions one clearly needed no initial orbit determination method (and they have even longer sidereal periods).* This leaves minor planets.

Let us review the solar system circa 1800. We knew of the planets of antiquity - Mercury, Venus, Earth, Mars, Jupiter, and Saturn.** By and large

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* Modern re-examination of the bases for the Adams/Leverrier and Lowell stories cast aspersions on the arithmetical validity of the work of these men. Notice no one doubts Newtonian gravity.

** The seven classical planets of antiquity were the above minus the Earth, plus the Moon and the Sun.

these lay in the plane of the ecliptic revolving about the Sun in nearly circular orbits (see Table I). Moreover the semi-major axes of the planets satisfied a progression discovered by Titius and recently published (in 1772) by Bode. As yet though there was no body for the fifth place. Presciently Laplace (1780) publishes his technique of initial orbit determination. The following year Herschel discovers the first new planet in the history of the world and it fits the above scheme very well. This plus earlier (i.e., the moons of Jupiter, etc.) discoveries shows the scientific community that the solar system does contain additional bodies. Believing in the numerology of the Titius-Bode relationship a search is organized for the missing planet. On the first night of the nineteenth century Piazzi discovers Ceres. He keeps the discovery to himself for three weeks, illness then forces him from the telescope after 41 nights (not 3!) of observing. Ceres beats the slow mails of the winter of 1801 to conjunction. Is this the missing planet and how shall we find it after conjunction?

I regard this as the first time an orbit was really needed. The stories one reads in contemporary astronomy books recounts that Gauss heard of the difficulty in September, invented his method of initial orbit determination in October, predicted (in November) a position for Ceres and von Zach (a member of the original search team; Piazzi was a potential member; Bode organized it) found it with 0°.5 of that position on January 1, 1802 (it was cloudy in December*). What better triumph can one ask of science? How far

* Some accounts have one clear night, December 7, during which von Zach "glimpsed" it. If so, this position obviously allow a differential correction of large weight.

TABLE I

PLANETARY DATA

Planet	Semi-Major Axis (A.U.)	Bode's Law Value (A.U.)	Eccentricity	Inclination
Mercury	0.387	0.4	0.206	7°00
Venus	0.723	0.7	0.007	3.39
Earth	1.000	1.0	0.017	0
Mars	1.524	1.6	0.093	1.85
(Ceres)	2.767	2.8	0.076	10.62
Jupiter	5.204	5.2	0.049	1.31
Saturn	9.580	10.0	0.051	2.49
Uranus	19.141	19.6	0.046	0.77

off could the prediction have been before we stopped trumpeting this epic? How did Gauss really do it? Before giving my answers to the last two questions, we need to complete the story. Olbers too found Ceres (on January 2) and then Pallas in the Spring. In 1804 Juno was discovered and in 1807 Olbers found Vesta. The next minor planet was discovered in 1845. Photographic searches were started in 1891

B. The Histor.cal Myth - Ceres

The above is the historical myth concerning the reacquisition of Ceres. Now to my questions above and then some speculations. I would guess that the errors could've been as large as 5° before the power of the tale pales. With perfect hindsight, and both theoretical and observational expertise in what I'm about to propose, I'd have done it as follows: The solar system lies in a plane and this new object is discovered near it. Hence the inclination is zero and the longitude of the ascending node is superfluous. All orbits are circular* so I'll assume zero for the eccentricity and the argument of perihelion is meaningless. I believe in Bode's law so I know the semi-major axis. Moreover the assumed semi-major axis correctly reproduces the observed angular speed. This leaves a single orbital element, the mean longitude (say) to fix. Finally, had I tried all of this for the just discovered Uranus it would've worked (look at Table I). I would've tried it for Ceres and since I'm temporarily Gauss, the inventor of least squares, I would've found an intelligent way to use Piazzi's 41 nights of data and perform a simple differential correction of the orbit.

* Note that of the inner planets Mercury has both the largest eccentricity and the highest inclination. Even its values aren't huge and it's the closest to the Sun.

We don't know how Gauss actually computed the orbit of Ceres or any of the other big four minor planets. We do know that however he did it, Gauss did not use the method published in his Theoria Motus in 1809 (reference 3). I quote at length, and in context, from the Preface to that work, first concerning Uranus:

"As soon as it was ascertained that the motion of the new planet, discovered in 1781, could not be reconciled with the parabolic hypothesis, astronomers undertook to adapt a circular orbit to it, which is a matter of simple and very easy calculation. By a happy accident the orbit of this planet had but a small eccentricity, in consequences of which the elements resulting from the circular hypothesis sufficed at least for an approximation of which could be based the determination of the elliptic elements. There was a concurrence of several other very favorable circumstances. For, the slow motion of the planet, and the very small inclination of the orbit to the plane of the ecliptic, not only rendered the calculations much more simple, and allowed the use of special methods not suited to other cases; but they removed the apprehension, lest the planet, lost in the rays of the sun, should subsequently elude the search of observers, (an apprehension which some astronomers might have felt, especially if its light had been less brilliant); so that the more accurate determination of the orbit might be safely deferred, until a selection could be made from observations more frequent and more remote, such as seemed best fitted for the end in view."

The next paragraph of the Preface discusses the general problem:

"Thus, in every case in which it was necessary to deduce the orbits of heavenly bodies from observations, there existed advantages not to be

despised, suggesting, or at any rate permitting, the application of special methods; of which advantages the chief one was, that by means of hypothetical assumptions an approximate knowledge of some elements could be obtained before the computation of the elliptic elements was commenced. Notwithstanding this, it seems somewhat strange that the general problem,--

To determine the orbit of a heavenly body, without any hypothetical assumption, from observations not embracing a great period of time, and not allowing a selection with a view to the application of special methods, was almost wholly neglected up to the beginning of the present century; or, at least, not treated by any one in a manner worthy of its importance; since it assuredly commended itself to mathematics by its difficulty and elegance, even if its great utility in practice were not apparent. An opinion had universally prevailed that a complete determination from observations embracing a short interval of time was impossible,--an ill-founded opinion,--for it is now clearly shown that the orbit of a heavenly body may be determined quite nearly from good observations embracing only a few days; and this without any hypothetical assumption."

Finally, Gauss on Gauss and Ceres:

"Some ideas occurred to me in the month of September in the year 1801, engaged at the time on a very different subject, which seemed to point to the solution of the great problem of which I have spoken. Under such circumstances we not unfrequently, for fear of being too much led away by an attractive investigation, suffer the associations of ideas, which, more attentively considered, might have proved most fruitful in results, to be lost from neglect. And the same fate might have befallen these conceptions, had they

not happily occurred at the most propitious moment for their preservation and encouragement that could have been selected. For just about this time the report of the new planet, discovered on the first day of January of that year with the telescope at Palermo, was the subject of universal conversation; and soon afterwards the observations made by that distinguished astronomer PIAZZI from the above date to the eleventh of February were published. Nowhere in the annals of astronomy do we meet with so great an opportunity, and a greater one could hardly be imagined, for showing most strikingly, the value of this problem, than in this crisis and urgent necessity, when all hopes of discovering in the heavens this planetary atom, among innumerable small stars after the lapse of nearly a year, rested solely upon a sufficiently approximate knowledge of its orbit to be based upon these very few observations. Could I ever have found a more seasonable opportunity to test the practical value of my conceptions, that now in employing them for the determination of the orbit of the planet Ceres, which during these forty-one days had described a geocentric arc of only three degrees, and after the lapse of a year must be looked for in a region of the heavens very remote from that in which it was last seen? This first application of the method was made in the month of October, 1801, and the first clear night, when the planet was sought* for as directed by the numbers deduced from it, restored the fugitive to observation. Three other new planets, subsequently discovered, furnished new opportunities for examining and verifying the efficiency and generality of the method.

Several astronomers wished me to publish the methods employed in these calculations immediately after the second discovery of Ceres; but many things--

*By de Zach, December 7, 1801.

other occupations, the desire of treating the subject more fully at some subsequent period, and, especially, the hope that a further prosecution of this investigation would raise various parts of the solution to a greater degree of generality, simplicity, and elegance,--prevented my complying at the time with these friendly solicitations. I was not disappointed in this expectation, and have no cause to regret the delay. For, the methods first employed have undergone so many and such great changes, that scarcely any trace of resemblance remains between the method in which the orbit of Ceres was first computed, and the form given in this work."

C. Modern Reality - Chiron

It should be instructive to review the published history of the slow moving object discovered by C. T. Kowall in 1977. It is now known as minor planet 2060 Chiron. It's preliminary designation was 1977UB.

The first International Astronomical Union Circular containing information about "Slow-moving Object Kowal" was Number 3129 (dated Nov. 4, 1977). It reported two accurate positions by Kowal (separated by 25 hours) and one approximate position by Gehrels from a photographic plate taken a week (Oct. 11) earlier. The motion was very slow and retrograde, at least one-third of that of a main belt asteroid. Presumably it was the strange motion that kept the Minor Planet Center from publishing (a potentially embarrassing) orbital element set. Four days later I.A.U. Circular No. 3130 reported two accurate positions from Gehrels (replacing his preliminary one) and two more positions from Mt. Palomar acquired on November third and fourth. An orbital element set accompanied this and it was labeled "extremely indeterminate". The important parameters are the eccentricity $e = 0.031$ and the

period P = 66.1 years. We are advised that this orbit was "selected so as to minimize the aphelion distance". As e is essentially zero this means minimizing the period.

Seven days later I.A.U. Circular No. 3134 reported another pair of observations (Nov. 9 and 10). It also contained the comment "that a near-circular orbit solution (cf. IAUC 3130) is still viable, but an ellipse of very high eccentricity is not". Very high is not defined (0.9 or 0.5?). Additional observations from mid-November are reported in Circulars Nos. 3140 and 3143. Finally, by the end of November the jig is up. Circular No. 3145 reports two observations from 1969 based on the work of J. G. Williams. A new orbital element set is also included, e = 0.37860, P = 50.70 yrs. Not very circular. Within another week pre-discovery images from 1952 and the early 1940's are reported (Circular No. 3147). What is not reported there (but is in reference 4) is the error in the predicted position -- 1°1 for the 1969 observations based on the original orbital element set, 0°25 for the 1952 observations based on the improved element set, and 0°5 for the 1941 points. Williams had at least 15 positions to use to deduce the orbit that allowed him to find the 1969 positions. Also not mentioned is that finding the short, faint, trail of Chiron is as much luck as celestial mechanics--the trail was marked on the 1941 plate <u>in</u> 1941 but subsequently ignored.

Finally, by mid-December, an observation from 1895 has been reported and yet another orbit produced by the process of differential correction. Now e = 0.378623 and P = 50.68 yrs. The small inclination has changed by 33% from its original value of i = 5°2 (to 6°9229) but the effect and importance of this are small. After the publication of this Circular (No. 3151) additional observations from 1940's and 1976 appear (Nos. 3156, 3215).

In my opinion it is clear that the deduction of a reasonable orbit for 2060 Chiron depended much more on modern communications and large scale (6° x 6°) photographic plates than it did on Carl Fredrich Gauss.

D. Modern Initial Orbit Determination

Modern initial orbit determination is concerned with rockets and artificial satellites. While optical observations (both passive and active) have and continue to be performed on these objects radar, sans doute is the premier observing technique. Radars give distance (radio detecting and ranging) and, distance determination is the essence of initial orbital determination by optical means.

In the only instance of which I am aware, when Gauss's method (really it was Gibb's 1888 refinement but that's a detail) was used on high altitude artificial satellites it failed (see reference 5). The causes of the failure were two-fold. The permissible range of validity of the method was exceeded and this is clearly no fault of the method. (Try and find a discussion of this point in a celestial mechanics book though.) The other reason is that the method doesn't work. It never did, it never has.

E. Outline of the Remainder

Section II reviews, in a more rigorous fashion than usual, the fundamentals of Laplacian orbit determination. Most solar system problems either have the Sun or the Earth as a force center and both of these cases are covered. This Section also presents an elucidation of the essential difference between the Laplacian and Gaussian forms of initial orbit determination. It concludes with a short discussion of refinements of the Laplacian technique.

Section III discusses polynomial smoothing, through quartics, in as general a manner as possible. Algebraic complexity rapidly overcomes universality though and further progress is made by assuming a uniform spacing in time for data acquisition. Ideally one would want the minimum variance analysis to dictate the frequency distribution of data acquisition. Unfortunately, the fulfillment of this ideal is either beyond my patience or simply impossible because the problem is intractable.

Section IV discusses new tests of the high quality data rich Laplacian method. It has been applied to an asteroid discovered by us (1982HS). This is in a very rare high inclination, high eccentricity orbit. We have also applied it to the original Earth-approaching minor planet (1862 Apollo), and a high inclination, high eccentricity, 2 rev/day artificial satellite.

The last Section discusses the problem of trigonometrical parallax determination for artifical satellites by non-simultaneous optical observations. This is a natural outgrowth of our work on smoothing a large number of rapidly acquired angles-only observations.

II. LAPLACE'S METHOD

In this Section I present Laplace's method from a heliocentric point of view for minor planets and geocentric point of view for artificial satellites. The last subsection stresses the essential mathematical differences between the Laplacian and Gaussian techniques.

A. Artificial Satellites

Let the observer's geodetic datum be (H,λ,ϕ') his height above the reference ellipsoid, his geocentric longitude, and his geocentric latitude. From these data we can compute the observer's geocentric distance ρ and the local sidereal time τ corresponding to the local solar time t (which is simply related to Universal Time; Ephemeris Time buffs will have to wait their turn). Thus, we know the geocentric location of the observer $\underline{\rho}$ and can calculate its derivatives with respect to t.

Consider now some artificial satellite whose geocentric location is $\underline{r}$ in the equatorial coordinate system. Its topocentric location $\underline{R}$ is related to $\underline{r}$ and $\underline{\rho}$ via

$$\underline{r} = \underline{R} + \underline{\rho} \tag{1}$$

We can measure the direction to the satellite. To stress this let us write

$$\underline{R} = R\underline{\ell} \tag{2}$$

Here $\underline{\ell}$ is the unit vector of topocentric direction cosines. We substitute (2) into (1) and differentiate twice with respect to t;

$$\underline{r} = R\underline{\ell} + \underline{\rho} \tag{3a}$$

$$\dot{\underline{r}} = \dot{R}\underline{\ell} + R\dot{\underline{\ell}} + \dot{\underline{\rho}} \tag{3b}$$

$$\ddot{\underline{r}} = \ddot{R}\underline{\ell} + 2\dot{R}\dot{\underline{\ell}} + R\ddot{\underline{\ell}} + \ddot{\underline{\rho}} \tag{3c}$$

Since the satellite does orbit the Earth,

$$\ddot{\underline{r}} = - GM_E \, \underline{r}/r^3 \tag{4}$$

where $r = |\underline{r}|$ and GM_E is the gravitational constant for the Earth. We replace the left hand side of Eq. (3c) with its equivalent in (4) to find that

$$\frac{-GM_E \underline{r}}{r^3} = \ddot{R}\underline{\ell} + 2\dot{R}\dot{\underline{\ell}} + R\ddot{\underline{\ell}} + \ddot{\underline{\rho}} \tag{5}$$

We can isolate R by finding a vector perpendicular to both $\underline{\ell}$ and $\dot{\underline{\ell}}$. Clearly $\underline{\ell} \times \dot{\underline{\ell}}$ will do and

$$R = \frac{-1}{(\underline{\ell} \times \dot{\underline{\ell}}) \cdot \ddot{\underline{\ell}}} \left[\frac{GM_E}{r^3} \; (\underline{\ell} \times \dot{\underline{\ell}}) \cdot \underline{r} + (\underline{\ell} \times \dot{\underline{\ell}}) \cdot \ddot{\underline{\rho}} \right] \tag{6}$$

Next we take the scalar product of Eq. (3a) with itself to obtain

$$r^2 = R^2 + 2R\underline{\ell} \cdot \underline{\rho} + \rho^2 \tag{7}$$

Finally, we replace $\underline{r}$ on the right hand side of (6) with its value from (3a), viz.

$$R = \frac{-1}{(\underline{\ell} \times \dot{\underline{\ell}}) \cdot \ddot{\underline{\ell}}} \left[\frac{GM_E}{r^3} \; (\underline{\ell} \times \dot{\underline{\ell}}) \cdot \underline{\rho} + (\underline{\ell} \times \dot{\underline{\ell}}) \cdot \ddot{\underline{\rho}} \right] \tag{8}$$

Equations (7) and (8) are two coupled, non-linear equations in the two unknowns r and R. They are equivalent to a single eighth-order polynomial in r.

If we step back from the algebra and look at the terms appearing in these two equations they are of three forms--quantities we know or can compute $(\rho, \dot{\rho}, \ddot{\rho})$, quantities we measure ($\underline{\ell}$), and quantities we need a way of calculating ($\dot{\underline{\ell}}$ and $\ddot{\underline{\ell}}$). The formal solution to the initial orbit determination process requires $\underline{r}$ and $\dot{\underline{r}}$ (or $\underline{R}$ and $\dot{\underline{R}}$). Thus we need an equation for $\dot{R}$. Clearly we may obtain one from Eq. (5) upon scalar multiplication by $\underline{\ell} \times \ddot{\underline{\ell}}$,

$$2\dot{R} = \frac{-1}{(\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \dot{\underline{\ell}}} \left[\frac{GM_E}{r^3} \; (\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \underline{r} + (\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \ddot{\underline{\rho}} \right]$$

or after replacement of $\underline{r}$ by $R\underline{\ell} + \underline{\rho}$,

$$2\dot{R} = \frac{-1}{(\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \dot{\underline{\ell}}} \left[\frac{GM_E}{r^3} \; (\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \underline{\rho} + (\underline{\ell} \times \ddot{\underline{\ell}}) \cdot \ddot{\underline{\rho}} \right] \tag{9}$$

The reader should note that these results are completely rigorous. I will discuss at length below (Section III) methods to obtain approximations to the topocentric angular velocity $\dot{\underline{\ell}}$ and the topocentric angular acceleration $\ddot{\underline{\ell}}$.

B. Asteroids

Let

$\underline{r}$ = heliocentric equatorial location of the asteroid

$\underline{R}_S$ = geocentric equatorial location of the Sun = (X_S, Y_S, Z_S)

$\underline{\rho}$ = geocentric location of observer = $\rho\underline{\ell}\,(\tau, \phi') = \rho(\xi, \eta, \zeta)$

$\underline{R}$ = topocentric equatorial location of the asteroid = $R\underline{\ell}\,(A, \Delta)$ = $R(\lambda, \mu, \nu)$

where the vector of direction cosines is given by

$$\underline{\ell}(\alpha,\delta) = (\cos\delta\cos\alpha,\ \cos\delta\sin\alpha,\ \sin\delta)$$

From Fig. 1 we see that

$$\underline{r} = -\underline{R}_S + \underline{\rho} + \underline{R} \tag{10}$$

The minor planet orbits the Sun. Ignoring the mass of the asteroid compared to that of the Sun ($=M_S$) and planetary perturbations, (G is the universal constant of gravitation)

$$\ddot{\underline{r}} = -GM_S\ \underline{r}/r^3 \tag{11}$$

Substituting Eq. (10) into Eq. (11) yields

$$\ddot{\underline{R}} + \ddot{\underline{\rho}} - \ddot{\underline{R}}_S = -GM_S\ (\underline{R} + \underline{\rho} - \underline{R}_S)/r^3 \tag{12}$$

But the Sun orbits the Earth and (approximately; neglecting the mass of the Moon, the geocentric distance of the Earth-Moon barycenter, and planetary perturbations again)

$$\ddot{\underline{R}}_S = -GM_S\ \underline{R}_S/R_S^3 \tag{13}$$

Thus

$$\ddot{\underline{R}} + GM_S\ \underline{R}/r^3 = -GM_S\underline{R}_S\ (1/R_S^3 - 1/r^3) - \ddot{\underline{\rho}} - GM_S\underline{\rho}\ /r^3 \tag{14}$$

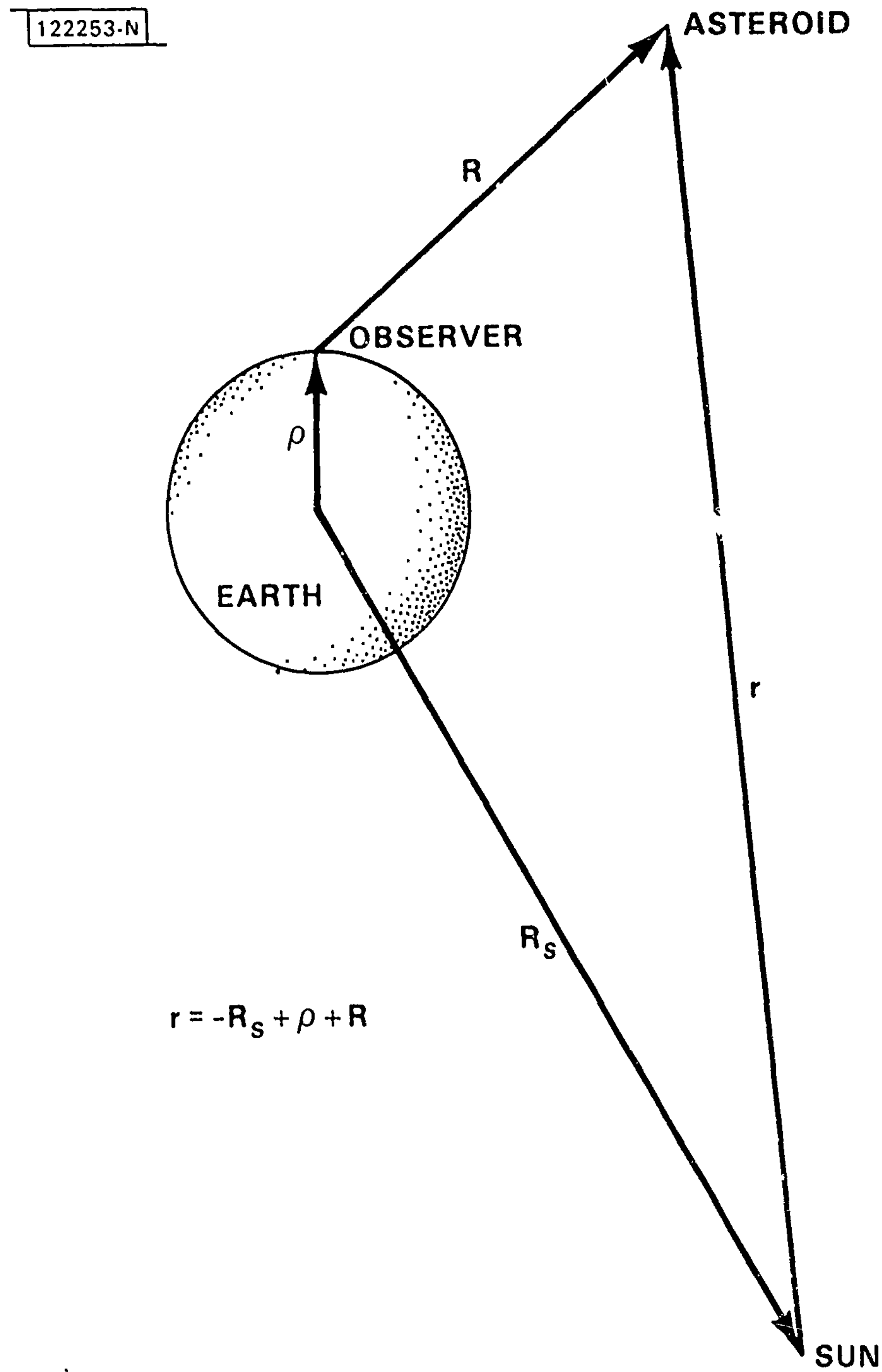

Fig. 1. Exaggerated diagram showing the location of the Sun, Earth, Observer, and asteroid.

Consider (say) the x component of this equation. As $d^2(R\lambda)/dt = \ddot{R}\lambda + 2\dot{R}\dot{\lambda} + R\ddot{\lambda}$ and ρ and ϕ' are constants, one finds

$$\lambda\ddot{R} + \dot{\lambda}(2\dot{R}) + (\ddot{\lambda} + GM_s\lambda/r^3)R = -GM_sX_s\ (1/R_s^3 - 1/r^3) + \rho\xi(\dot{\tau}^2 - GM_s/r^3) \tag{15}$$

If we write down the corresponding y and z equations then we regard this as a system of three linear, inhomogenous equations in the three unknowns $\ddot{R}$, $2\dot{R}$, and R. The determinant of the system, D, is given by

$$D = \begin{vmatrix} \lambda & \dot{\lambda} & \ddot{\lambda} + GM_s\lambda/r^3 \\ \mu & \dot{\mu} & \ddot{\mu} + GM_s\mu/r^3 \\ \nu & \dot{\nu} & \ddot{\nu} + GM_s\nu/r^3 \end{vmatrix} = \begin{vmatrix} \lambda & \dot{\lambda} & \ddot{\lambda} \\ \mu & \dot{\mu} & \ddot{\mu} \\ \nu & \dot{\nu} & \ddot{\nu} \end{vmatrix} \tag{16}$$

The solution for R is

$$R = D_1'/D \tag{17}$$

with

$$D_1' = \begin{vmatrix} \lambda & \dot{\lambda} & -GM_sX_s\ (1/R_s^3 - 1/r^3) + \rho\xi(\dot{\tau}^2 - GM_s/r^3) \\ \mu & \dot{\mu} & -GM_sY_s\ (1/R_s^3 - 1/r^3) + \rho\eta(\dot{\tau}^2 - GM_s/r^3) \\ \nu & \dot{\nu} & -GM_sZ_s\ (1/R_s^3 - 1/r^3) + \rho\zeta(o - GM_s/r^3) \end{vmatrix} \tag{18}$$

If $\rho = 0$ we can simplify D_1' to

$$D_1'\ (\rho = 0) = -GM_s\ (1/R_s^3 - 1/r^3) \begin{vmatrix} \lambda & \dot{\lambda} & X_s \\ \mu & \dot{\mu} & Y_s \\ \nu & \dot{\nu} & Z_s \end{vmatrix} \equiv D_1 \tag{19}$$

Note that if one knows the topocentric direction cosines [eg. (λ,μ,ν)], the topocentric angular velocity [eg. $(\dot{\lambda},\dot{\mu},\dot{\nu})$], and the topocentric angular acceleration [eg. $(\ddot{\lambda},\ddot{\mu},\ddot{\nu})$] then the formal solution for R in Eq. (17) is really an expression for R(r) as the observer's location, velocity, and acceleration as well as the solar location, velocity, and acceleration are known. Hence, to solve the problem we need one more relationship between R and r. To obtain it we square the basic geometrical relationship Eq. (10). The result is

$$r^2 = R^2 + 2\rho R\cos Z - 2RR_s\cos\psi + \rho^2 + R_s^2 - 2\underline{\rho} \cdot \underline{R}_s \tag{20}$$

where Z is the topocentric zenith distance of the minor planet [$\underline{\rho}\cdot\underline{R} = \rho R\cos Z$], and $\cos\psi = (\lambda X_s + \mu Y_s + \nu Z_s)/R_s$. This completes the solution of the problem. We have two equations in two unknowns. One expresses the physics, the other the geometry. If we take $\rho = 0$, then they become

$$R = (D_1/D)\,[1/R_s^3 - 1/r^3]$$

$$r^2 = R^2 - 2R\,R_s\cos\psi + R_s^2 \tag{21}$$

The form of these is classic in angles only initial orbit determination as are the facts that the coupled system is equivalent to a single polynomial equation of the eighth degree,

$$s^8 - (a^2 - 2a\cos\psi + 1)s^6 + 2a(a - \cos\psi)s^3 - a^2 = 0 \tag{22}$$

[where $s = r/R_s$ and $a = D_1/DR_s^4$] and that s = 1, R = 0 is a solution. The latter represents the Earth. Note that the explicit inclusion of the diurnal parallax removes this degeneracy--a useful point for Earth-approaching asteroids.

Let me summarize: If one knows one's location, velocity, and acceleration relative to the center of the Earth (as one can), and if one knows the Earth's (equivalently the Sun's) location, velocity, and acceleration (again, as one can), and if one knows the minor planet's position, angular velocity, and angular acceleration (see below), then one can compute the topocentric distance of the asteroid and its topocentric radial velocity. The first statement follows since R is a solution of the coupled R(r) set, Eq. (21). One finds $\dot{R}$ by returning to Eq. (15) and solving for $2\dot{R}$, viz.

$$2\dot{R} = D_2'/D \tag{23}$$

where

$$D_2' = \begin{vmatrix} \lambda - GM_sX_s\ (1/R_s^3 - 1/r^3) + \rho\xi(\dot{\tau}^2 - GM_s/r^3) & \ddot{\lambda} \\ \mu - GM_sY_s\ (1/R^3 - 1/r^3) + \rho\eta(\dot{\tau}^2 - GM_s/r^3) & \ddot{\mu} \\ \nu - GM_sZ_s\ (1/R^3 - 1/r^3) + \rho\zeta(0 - GM_s/r^3) & \ddot{\nu} \end{vmatrix} \tag{24}$$

Note that

$$D_2'\ (\rho = 0) = -\ GM_s(1/R_s^3 - 1/r^3) \begin{vmatrix} \lambda & X_s & \ddot{\lambda} \\ \mu & Y_s & \ddot{\mu} \\ \nu & Z_s & \ddot{\nu} \end{vmatrix} = D_2 \tag{25}$$

Clearly once a complete topocentric specification of location and velocity is available (coupled with the preserved ancillary information $\underline{\rho}$, $\dot{\underline{\rho}}$, $\underline{R}_s$, and $\dot{\underline{R}}_s$) one can produce a corresponding heliocentric set. Going from the data $\underline{r}$ and $\dot{\underline{r}}$ at some time $t = t_0$ to the orbital element set is a straightforward algebra problem.

The essential points are these - i) there is no appreciable neglect of physics, and ii) the geometry and physics are enforced at a single instant of time. In practice we relax the proviso that no mathematical approximations are made by numerically differentiating both $\underline{R}_s$ and $\underline{\ell}(A,\Delta) = (\lambda,\mu,\nu)$ to obtain $\dot{\underline{R}}_s$, $\dot{\underline{\ell}}$, and $\ddot{\underline{\ell}}$. Note that this enforces the geometrical constraint at more than one point while still enforcing the physics at a single instant of time. Note too that while numerical differentiation may well be inaccurate, it is neither theoretically impossible, nor forbidden, nor can the time span of the observations preclude orbit determination.

How does one obtain $\dot{\underline{\ell}}$ and $\ddot{\underline{\ell}}$? At a minimum one needs three values of $\underline{\ell}$ to determine $\ddot{\underline{\ell}}$. Using only three sets amounts to a trick as far as I'm concerned. Furthermore, on slow moving objects such as asteroids (~0°5-2°/day for the really fast ones) numerical differentiation of three observations is criminal. On the other hand, smoothing a dozen or two observations over a night and then analytically differentiating might work.

C. The Gaussian Difference

An obvious method to obtain estimates for $\dot{\underline{\ell}}$ and $\ddot{\underline{\ell}}$ is to numerically differentiate an interpolating polynomial obtained from a set of $\underline{\ell}$ values. If we use three such values then we both satisfy our ego (since we know that only three are necessary) and are just able to compute an approximation for both $\dot{\underline{\ell}}$ and $\ddot{\underline{\ell}}$. Clearly with observational data of low accuracy acquired on slowly moving objects the inherently unstable process of central difference approximations for second derivatives will produce a value for $\ddot{\underline{\ell}}$ of little utility or resemblance to reality. Thus, in the past, in practice, Laplace's method failed.

What Gauss did that was different was to exploit the fact that the orbit lies in a plane. Then, for three points at t_1, t_2 and t_3 one can write

$$a\underline{r}_1 + b\underline{r}_2 + c\underline{r}_3 = \underline{0}$$

since the three three-dimensional location vectors are linearly dependent. From the fact that the vector cross product yields an area and the known properties of central force motion, one can derive formulas for a, b, and c. The essential step that makes the computation feasible is the use of power series (vs. numerical differentiation) to express $\underline{r}_1$ and $\underline{r}_3$ in terms of $\underline{r}_2$. We call these series the f and g series. As discussed at length in reference 6 (but nowhere else in the entire celestial mechanics literature!) the radius of convergence of these series rapidly approaches 0 as the eccentricity of the orbit approaches unity. Therefore, one may be a priori forbidden to use the Gaussian technique and not know it. Thus, the central element of contrariness between the methods of Laplace and Gauss lies in the nature of an approximation. Laplace's method forces one to make a <u>numerical</u> approximation in order to calculate the angular velocity and the angular acceleration of the unknown. Note that the restriction to three observations is merely minimal. Gauss's method forces one to make an <u>analytical</u> approximation in order to calculate the ratio of the area of a sector to its associated triangle in Keplerian motion. Again the use of only three sets of angular measurements rests on minimality and not necessity. Note too that while a particular numerical differentiation may be more or less accurate [and can be arbitrarily refined <u>without</u> expanding the observational time span (but by increasing the data acquisition rate)], Gauss's approximation may fail catastrophically without notice or means of redress ex post facto.

D. Refinements

Most of the refinements of the Laplacian method that I have seen amount to a form of differential correction using the f and g series. I will not advocate any use of the f and g series in initial orbit determination and reject all of them. I would recommend that the next step be a differential correction process at the individual instants of observation.*

The principal refinement of the Gaussian technique was developed by Gibbs in 1888. It allows one to include additional terms in the f and g series. As the problem is convergence, not truncation, little practical advancement was accomplished. Actually, since the newly included terms are radial velocity (e.g. eccentricity) dependent, one probably gains a false sense of security when using Gibb's refinement of Gauss's technique. For if the eccentricity is small, then so is the radial velocity and the additional terms are unimportant. Moreover the radius of convergence of the f and g series is an appreciable fraction of the period. On the other hand when the eccentricity is large, the radius of convergence is shorter in duration but one might feel better using Gibb's advance because eccentricity dependent terms are being included. One should feel better if the problem were a truncation difficulty rather than a convergence failure.

* As the next Section discusses, all of the individual observations should be combined to yield a single set of estimators for $\underline{\ell}$, $\dot{\underline{\ell}}$ and $\ddot{\underline{\ell}}$.

III. DATA SMOOTHING

A. The Fundamental Decisions for Asteroids

The fundamental decisions concern the time interval over which the smoothing is to be performed and the form of the smoothing function. In both the fast moving minor planet and artificial satellite cases we rely on Weierstrass's approximation theorem and use a polynomial form. The degree of the polynomial is not so quickly decided upon and will be discussed in detail below. In addition, while one would clearly smooth at most over a few hours for an artificial satellite, one might do so for as long as a week (or be forced to do so for such a duration) for an asteroid. Hence let us first turn our attention to the relative advantages of night-to-night preliminary smoothing for a fast moving minor planet versus a simultaneous fit of several nights observations. While doing this we must keep in mind that what we want are the best estimators for R and $\dot{R}$. Considering the transcendental nature of the dependence of R and $\dot{R}$ on the observations [cf. Eqs. (17) and (23)] we settle for the best estimators of A, $\dot{A}$ and $\ddot{A}$ and for Δ, $\dot{\Delta}$, and $\ddot{\Delta}$ (if possible to do so simultaneously).

At first glance the principal advantage of a nightly reduction of the observations is that a low order polynomial will do. On the other hand, since the total observing time is not a priori set and there may be an order of magnitude difference in the angular speeds of the asteroids we observe, a single polynomial may well not fit all cases of interest. The primary drawback of night to night reductions is the lack of a rigorous, clearly beneficial method of combination to yield position, angular velocity, and angular acceleration at a simultaneous epoch.

Let us first try and decide upon the appropriate order of the smoothing polynomial for an individual night's observations. A fortiori nothing less than a linear model will do -- even if one has a "single" observation.* As the general premise of this work is that of an abundance of data, the question of the highest order polynomial to be used must be addressed. Our analysis of the physics and geometry is complete except, necessarily, for planetary aberration. Therefore, since planetary aberration can not vary appreciably except in the most unusual of circumstances (e.g. an asteroid whose relative motion is nearly radial), an excessively high order polynomial is not necessary to absorb unmodeled effects. Thus, considering our quarry, the inclusion of fifth order terms seems superfluous, quadratic minimally sufficient. Only for the fastest asteroids observed over the longer nights might the quintic be appropriate. In general the results from the quartic fit can be used to check those from the cubic fit.

Another digression -- I am aware of the fact that I've just argued myself into the position of (theoretically) being able to determine an orbit based on an extremely short time span. All Laplace's method requires are direction cosines, their first and their second derivatives. Once the smoothing polynomial is second order these are all computable. However, a

* I must now digress for one never acquires a single observation of an artificial satellite or of an asteroid in a search mode. Consider a photographic search first. One discriminates the object because of its motion--a trail is left on the photographic plate (exposed with the telescope in sidereal drive so that the stars are held fixed) marking the passage of the object. Hence, even if this is the only record one can (and does in extreme cases) independently measure the endpoints of the trail to deduce two positions. Streak formation by electronic means is analogous as is forming a broken image by chopping with a rotating shutter. Hence, there are always at least two observations.

much better description of what I know is estimators for the direction cosines, their rates of change, and their accelerations. I also have estimates of the variance of these quantities which should preclude any premature attempt at initial orbital determination.

Ideally one would prefer to use some objective criteria for ascertaining the correct number and type of terms to include in the interpolating polynomial. Tests of significance, based on the F-test, can be constructed but lack a rigorous logical basis. Hence I would rely on experience and judgment to determine the correct degree of the interpolating polynomial and eschew apparently formal procedures of dubious value.

Finally each night is likely to be different. Some will be completely cloudy, some cloudy in only the first half, some clear, etc. Therefore, while quartic or quintic might be appropriate for the perigee passage of an Earth-approaching asteroid observed near the winter solstice, a quadratic fit the next evening (which is almost completely clouded out) would suffice. One cannot afford to lose this element of flexibility. So, after observing my minor planet for some number of nights I have, for each partially clear night, a position, probably a good angular velocity, and likely an estimate for the angular acceleration. In addition, the results of a given night's fit may not be simultaneously epoched (to obtain the minimum variance estimator, see below). From this inhomogenous and incomplete set of intermediate reductions I must now deduce the values for A, $\dot{A}$ and $\ddot{A}$, and for Δ, $\dot{\Delta}$, and $\ddot{\Delta}$ to begin the calculations of Laplace's method.

How? How indeed. I know of no theoretically sound method of combination that will unambiguously produce, in some well-defined and meaningful

statistical sense, "best" estimates for a position, angular velocity, and angular acceleration (all at one epoch). I can concoct a large variety of apparently reasonable procedures to do this. In order to quantitatively discuss their relative merits I need the results of subsection C below. Hence the remainder of this discussion is postponed to subsection D.

B. Artificial Satellites

In general the observing span should be as short as possible commensurate with obtaining good estimates for the desired quantities. The order of the polynomial necessary to do this depends on the topocentric appearance of the orbit -- contrast two circular orbits with the same periods but one with an inclination of zero and the other with an inclination of > 30°. The variety of combinations of orbital element sets and geographical circumstances are too numerous for any but the most general rules of thumb. It also becomes imperative that not only is the set of observations large but dense -- performing an observation must be rapid enough that high order polynomial fits are necessary because of the duration of the observing span. In any case, and the one to be dealt with in practice, we need at least a quadratic form and a quintic would be excessive. Finally, one can easily visualize situations wherein the order of the right ascension (or azimuth) polynomial is different from the declination (or altitude) polynomial. Experience and wisdom are needed in general in the artificial satellite case.

C. Polynomial Forms

A start on this subject at Lincoln Laboratory was made in a similar context (see references 7, 8). The problem is the following: given a number of observations of a quantity $x(t)$ and their associated times, say $\{x_n, t_n\}$

where $x_n = x(t_n)$. Let the weight of observation number n be w_n. Find, using polynomial forms to model x(t), the best estimates for x, dx/dt, and d^2x/dt^2 at some epoch t'.

Ideally one would solve the minimization problem via a maximum likelihood technique, form estimators for the parameters of the polynomial, deduce estimators and weights for x, $\dot{x}$, and $\ddot{x}$ at t' and then require that these quantities be evaluated at a time (=t') such that their variances are absolute minima. This may or may not yield a coeval set for x, $\dot{x}$ and $\ddot{x}$. Note that the search for the minimum variance estimators requires functional differentiation with respect to the probability distribution of performing an observation. In general this problem is intractable and I make two reasonable simplifying assumptions to speed the analysis (rather what's left of it). The first assumption is that the observations are executed, in time, symmetrically about $\langle t\rangle$.

$$\langle t\rangle = \sum_n w_n t_n \Big/ \sum_m w_m \tag{26}$$

Hence, for all odd k

$$\sum_n w_n \tau_n^k = 0 \quad , \quad \tau_n = t_n - \langle t\rangle$$

The second, much more restrictive assumption is that the observations are equally spaced with interval T,

$$\tau_n = nT \qquad n = -N, -N+1, \ldots, N \tag{27}$$

and have equal weights $w_n = w \;\; \forall n \in [-N, N]$. These two assumptions are

unnecessary when treating a zero'th order and first degree polynomial form for $x(t)$. Of course such forms don't yield an interesting estimator for $\ddot{X}$.

We shall need the sum S_K,

$$S_k = \sum_{n=-N}^{+N} w_n \tau_n^k \tag{28}$$

$$= \begin{cases} (2N + 1)w & k = 0 \\ 0 & k \text{ odd} \\ 2wT^k \sum_{n=1}^{N} n^k & k \text{ even} \end{cases}$$

In particular

$$S_0 = (2N + 1)w$$

$$S_2 = 2wT^2N(N + 1)(2N + 1)/6$$

$$S_4 = 2wT^4N(N + 1)(2N + 1)(3N^2 + 3N - 1)/30$$

$$S_6 = 2wT^6N(N + 1)(2N + 1)(3N^4 + 6N^3 - 3N + 1)/42$$

Note that as $N \to \infty$

$$\sum_{n=-N}^{+N} w_n \tau_n^k \to \frac{w}{T} \int_{-NT}^{NT} u^k du = \begin{cases} \frac{2w}{T} \frac{(NT)^{k+1}}{k+1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Note too that for large N it matters little whether there were 2N +1 or 2N observations.

It seems best to present the zero'th order and first order polynomial forms analysis in enough detail to allow for a straightforward reproduction by the reader and then present the quadratic, and cubic results in an abbreviated form using a similar format.

1. Constant

The model is

$$x(t) = a \tag{29}$$

and the data is $\{x_n, t_n\}$, $n = -N, -N + 1, \ldots, N$ where $x_n = x(t_n)$. The n'th datum has a weight w_n. We minimize the sum of the square of the residuals

$$R = \sum_{n=-N}^{+N} w_n \, [x_n - a]^2 \tag{30}$$

with respect to the parameter a. This leads to the normal equations

$$MA = D \tag{31}$$

where $M = (S_0)$ is the matrix of the normal equations, $A = (a)$ is the vector of polynomial parameters, and D is the vector of observations $D = (\sum w_n x_n)$. By inverting M we find our estimators for the elements of A,

$$\hat{a} = \sum_n w_n x_n \Big/ \sum_m w_m \tag{32}$$

and along the diagonal of M^{-1} the estimators for the variances of the elements of A,

$$\text{var}\ (\hat{a}) = 1/\sum_n w_n \tag{33}$$

The off-diagonal elements of M^{-1} provide estimators for the covariances of the elements of A (=0 here).

We now use our model to propogate x and obtain an estimator for x(t) at any time, $\hat{x}(t)$. Next we calculate an estimate for the variance of $\hat{x}(t)$ via (in this simple case)

$$\text{var}[\hat{x}(t)] = (\partial\hat{x}(t)/\partial\hat{a})^2\ \text{var}(\hat{a}) = \frac{1}{\sum_n w_n} \tag{34}$$

2. Linear

The model is

$$x(t) = a + b\ (t - t_0) \tag{35}$$

where t_0 is an arbitrary epoch. We let

$$\tau = t - t_0$$

Form

$$R = \sum_n w_n\ [x_n - a - b\tau_n]^2$$

and minimize R with respect to a and b. This leads to

$$M = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} , \quad A = \begin{pmatrix} a \\ b \end{pmatrix} , \quad D = \begin{pmatrix} \sum w_n x_n \\ \sum w_n x_n \tau_n \end{pmatrix} \tag{36}$$

Also,

$$M^{-1} = |M|^{-1} \begin{pmatrix} S_2 & -S_1 \\ -S_1 & S_0 \end{pmatrix} \tag{37}$$

where $|M| = \det (M) = S_0 S_2 - S_1^2$. Again $A = M^{-1}D$ and

$$\begin{aligned} &\mathrm{var}\ (\hat{a}) = S_2/|M| \qquad \mathrm{cov}\ (\hat{a}, \hat{b}) = -S_1/|M| \\ &\mathrm{var}\ (\hat{b}) = S_0/|M| \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} \mathrm{var}\ [\hat{x}(t)] = &= (\partial\hat{x}/\partial\hat{a})^2\ \mathrm{var}\ (\hat{a}) + (\partial\hat{x}/\partial\hat{b})^2\ \mathrm{var}\ (\hat{b}) + \\ &+ 2(\partial\hat{x}/\partial\hat{a})(\partial\hat{x}/\partial\hat{b})\ \mathrm{cov}\ (\hat{a},\hat{b}) \\ &= (\tau^2 S_0 - 2\tau S_1 + S_2)/|M| \end{aligned}$$

We now ask at what value of $\tau = t - t_0$ is var $[\hat{x}(t)]$ a minimum? We find that $\partial \mathrm{var}\ [\hat{x}(t)]/\partial t = 0$ iff

$$\tau = \tau_e \equiv S_1/S_0 = \langle t \rangle - t_0$$

that is, the variance of $\hat{x}(t)$ is an extremum at $t = \langle t \rangle$. Moreover $\partial^2 \mathrm{var}[\hat{x}(t)]/\partial t^2 = 2S_0/|M| \geq 0$ so it is a minimum.* The value of var $[\hat{x}(t)]$ at $t = \langle t \rangle$is

* That $|M| \geq 0$ follows from an application of the Cauchy inequality.

$1/S_0$, just as in the previous case. The variance $\hat{\dot{x}}(t)$ is equal to var $(\hat{b})$ = $S_0/|M|$. Now choose t_0 = <t> then $S_1 = 0$ and cov $(\hat{a},\hat{b}) = 0$. Since $|M| = S_0 S_2$ now the general expression for var $[\hat{x}(t)]$ is

$$\mathrm{var}[\hat{x}(t)] = 1/S_0 + \tau^2/S_2$$

which clearly shows that $\tau = 0$ (t = <t>) is that instant when var $[\hat{x}(t)]$ is a minimum. Further note that min var $[\hat{x}(t)]$ is independent of the distribution in time of the observations. For $\dot{x}$ the situation is different, var $[\hat{\dot{x}}(t)] = 1/S_2$ and we want the largest possible spread of observing instants to minimize this quantity.

3. Quadratic

The model is

$$x(t) = a + b\tau + c\tau^2 \quad , \tau = t - \langle t\rangle \tag{39}$$

The normal equations are MA = D,

$$M = \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix} , \quad A = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad , \quad D = \begin{pmatrix} \Sigma w_n x_n \\ \Sigma w_n x_n \tau_n \\ \Sigma w_n x_n \tau_n^2 \end{pmatrix} \tag{40}$$

and

$$M^{-1} = |M|^{-1} \begin{pmatrix} S_2 S_4 - S_3^2 & S_2 S_3 - S_1 S_4 & S_1 S_3 - S_2^2 \\ S_2 S_3 - S_1 S_4 & S_0 S_4 - S_2^2 & S_1 S_2 - S_0 S_3 \\ S_1 S_3 - S_2^2 & S_1 S_2 - S_0 S_3 & S_0 S_2 - S_1^2 \end{pmatrix} \tag{41}$$

where $|M| = S_0 S_2 S_4 + 2 S_1 S_2 S_3 - S_2^3 - S_0 S_3^2 - S_4 S_1^2$. Hence,

$$\mathrm{var}\,(\hat{a}) = (S_2S_4 - S_3^2)/|M| \;,\; \mathrm{cov}(\hat{a},\hat{b}) = (S_2S_3 - S_1S_4)/|M|$$
$$\mathrm{var}\,(\hat{b}) = (S_0S_4 - S_2^2)/|M| \;,\; \mathrm{cov}(\hat{b},\hat{c}) = (S_1S_2 - S_0S_3)/|M|$$
$$\mathrm{var}\,(\hat{c}) = (S_0S_2 - S_1^2)/|M| \;,\; \mathrm{cov}(\hat{a},\hat{c}) = (S_1S_3 - S_2S^2)/|M| \tag{42}$$

So,

$$|M|\,\mathrm{var}\,[\hat{x}(t)] = S_2S_4 - S_3^2 + 2(S_2S_3 - S_1S_4)\tau + (S_0S_4 + 2S_1S_3 - 3S_2^2)\tau^2$$
$$+\, 2(S_1S_2 - S_0S_3)\tau^3 + (S_0S_2 - S_1^2)\tau^4$$

Note though that our choice of t_0 = <t> renders S_1 nil.

To search for that value of τ which makes the variance of $\hat{x}(t)$ a minimum we need to solve a cubic (obtained by setting $S_1 = 0$ in the above quartic and then differentiating it with respect to t). Despite the fact that cubics are algebraically soluble in closed form, I expect little advance of knowledge here. Introducing the assumption of symmetry ($S_3 = 0$ too) makes the above quartic a quadratic in τ^2 and

$$|M|\,\frac{\partial\,\mathrm{var}\,[\hat{x}(t)]}{\partial t} = 2(S_0S_4 - 3S_2^2)\tau + 4S_0S_2\tau^3$$

the right-hand side vanishes if $\tau = 0$ or $\tau = \tau_{\pm}$ where

$$\tau_{\pm} \equiv \pm \left(\frac{3S_2^2 - S_0S_4}{2S_0S_2}\right)^{1/2}$$

We can also compute that

$$\frac{|M|}{(3S_2^2 - S_0S_4)} \frac{\partial^2 \mathrm{var}\,[\hat{x}(t)]}{\partial\, t^{\,2}} = \begin{cases} -2 & \tau = 0 \\ +4 & \tau = \tau_\pm \end{cases}$$

There are three possibilities:

$3S_2^2 - S_0S_4 < 0$ and $\tau_\pm$ are not real. Then $\tau = 0$ is the single absolute minimum of the variance of $\hat{x}(t)$.

$3S_2^2 - S_0S_4 > 0$ and $\tau_\pm$ are both real. Then $\tau = 0$ is a relative maximum for var $[\hat{x}(t)]$ and the absolute minima of var $[\hat{x}(t)]$ occurs at $\tau = \tau_\pm$.

$3S_2^2 - S_0S_4 = 0$ and $\tau_\pm = 0$. Now $\tau = 0$ is the single absolute minimum for var $[\hat{x}(t)]$ as well as a point of inflection.

There is one last point. Suppose that $\tau_\pm$ are real. In general

$$|M|\ \mathrm{var}\ [\hat{x}(0)] = S_2S_4$$

and

$$|M|\ \mathrm{var}\ [\hat{x}(\tau_\pm)] = S_2S_4 - (3S_2^2 - S_0S_4)^2/(4S_0S_2)$$

showing the decrease in the variance of $\hat{x}(t)$ from its value at $\tau = 0$ ($t = \langle t \rangle$).

Now let us turn to a general analysis of the variance of $\hat{\dot{x}}(t) = \hat{b} + 2\,\hat{c}\,\tau$. We have

$$|M|\ \mathrm{var}\ [\hat{\dot{x}}(t)] = S_0S_4 - S_2^2 + 4\tau(S_1S_2 - S_0S_3) + 4(S_0S_2 - S_1^2)\tau^2$$

and $\qquad \partial \text{var}\,[\hat{x}(t)]/\partial t$ vanishes iff

$$\tau = \dot{\tau}_e = \frac{S_0\, S_3 - S_1\, S_2}{2(S_0 S_2 - S_1^2)}$$

Furthermore at this value of $\tau, |M| \;\; \partial^2 \text{ var } [\hat{\dot{x}}(t)]/\partial t^2 = 8\,(S_0S_2 - S_1^2)$ so that this represents a minimum (the proof of the positiveness of the second derivative of $\hat{x}(t)$ with respect to t follows from a double application of the Cauchy inequality). Once we introduce the symmetry assumption this value of τ is just 0 and

$$\text{min var } [\hat{x}(t)] = 1/S_2$$

as in the linear model. In general $\hat{\ddot{x}}(t) = 2\,\hat{c}$ so var $[\hat{\ddot{x}}(t)] = \dfrac{4(S_0S_2 - S_1^2)}{|M|} \geq 0$.

Now let us introduce the uniform separation assumption into the above results. Then

$$\tau_{\pm}^2 = (2N^2 + 2N + 1)T^2/10$$

$$\text{var } [\hat{x}(0)] \rightarrow \frac{9(1 - 1/2\,N)}{8Nw}$$

$$\text{min var } [\hat{\dot{x}}(t)] = \frac{3(1 - 3/2\,N)}{8wN^3T^2}$$

$$\text{var } [\hat{\ddot{x}}(t)] = \frac{45(1 - 5/2\,N)}{2wN^5T^4}$$

and the reduction in the variance of $\hat{x}(t)$ from $\tau = 0$ to $\tau = \tau_{\pm}$ is 1/5 of its value.

4. Cubic

The model is

$$x = a + b\tau + c\tau^2 + d\tau^3 \quad , \qquad \tau = t - \langle t\rangle \tag{43}$$

Immediately introduce the symmetry assumption so that in MA = D

$$M = \begin{pmatrix} S_0 & 0 & S_2 & 0 \\ 0 & S_2 & 0 & S_4 \\ S_2 & 0 & S_4 & 0 \\ 0 & S_4 & 0 & S_6 \end{pmatrix} , \quad A = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} , \quad D = \begin{pmatrix} \Sigma w_n x_n \\ \Sigma w_n x_n \tau_n \\ \Sigma w_n x_n \tau_n^2 \\ \Sigma w_n x_n \tau_n^3 \end{pmatrix} \tag{44}$$

Here,

$$M^{-1} = |M|^{-1} \begin{pmatrix} S_4 m' & 0 & -S_2 m' & 0 \\ 0 & S_6 m & 0 & -S_4 m \\ -S_2 m' & 0 & S_0 m' & 0 \\ 0 & -S_4 m & 0 & S_2 m \end{pmatrix} \tag{45}$$

where

$$m = S_0 S_4 - S_2^2, \ m' = S_2 S_0 - S_4^2, \text{ and } |M| = mm'. \tag{46}$$

One finds that

$$|M| \operatorname{var}[\hat{x}(t)] = S_4 m' + (S_6 m - 2S_2 m')\tau^2 + (S_0 m' - 2S_4 m)\tau^4 + mS_2\tau^6$$

and the first derivative of $\hat{x}(t)$ with respect to τ vanishes if $\tau = 0$ or $\tau = \pm\tau_\pm$,

$$\tau^2_\pm \equiv \frac{-(S_0m' - 2S_4m) \pm [(S_0m' - 2S_4m)^2 - 3mS_2\ (S_6m - 2S_2m')]^{1/2}}{2S_2m}$$

At $\tau = 0$

$$|M|\ \frac{\partial^2 \text{ var } [x(t)]}{\partial\tau^2} = 2(S_6m - 2S_2m')$$

while at $\tau_\pm$ the same quantity is (it only depends on $|\tau_\pm|$)

$$\pm 8\tau^2_\pm\ [(S_0m' - 2S_4m)^2 - 3mS_2\ (S_6m - 2S_2m')]^{1/2}$$

Clearly either the radical surd is real or it's not. If it's not real then neither of $\tau^2_\pm$ are real and $\tau = 0$ provides the minimum of the variance of $\hat{x}(t)$ (and $S_6m - 2S_2m' > 0$). If the radical surd is real and only $\tau^2_+ > 0$ ($\tau^2_- < 0$), then $\tau = \pm\tau_+$ are the instants of absolute minima for var $[\hat{x}(t)]$ while at $\tau = 0$ it has a relative maximum. Finally if the radical surd is real and both τ^2_+ and τ^2_- are non-negative, then $\pm\tau_+$ are local minima, $\pm\tau_-$ are a local maxima, and 0 is a local minimum.

Let us continue this discussion by introducing the assumption of equal spacing in τ. Furthermore assume that N is large.

Then,

$$m \rightarrow \frac{16w^2T^4N^6}{45}(1 + 3/N)$$

$$m' \rightarrow \frac{16w^2T^8N^{10}}{525}(1 + 5/N)$$

and the discriminant for $\tau_{\pm}^2$ is >0. In fact,

$$\tau_{\pm}^2 = \frac{3T^2N^2}{7}(1 + 1/N) \quad , \qquad \tau_{-}^2 = \frac{T^2N^2}{5}(1 + 1/N)$$

We also have

$$|M| \text{ var } [\hat{x}(0)] = S_4 m' \rightarrow \frac{32w^3T^{12}N^{15}}{2625}(1 + 15/2N)$$

and

$$|M| \text{ var } [\hat{x}(\pm\tau_+)] = \left(\frac{15}{49}\right) \ |M| \text{ var } [\hat{x}(0)]$$

while

$$|M| \text{ var } [\hat{x}(\pm\tau_-)] = \left(\frac{19}{45}\right) \ |M| \text{ var } [\hat{x}(0)]$$

Figure 2 shows the normalized variance of $\hat{x}(t)$ in units of $|\tau|/NT$.

Now we can turn to the analysis of the variance of $\hat{\dot{x}}(t) = \hat{b} + 2\hat{c}\tau + 3\hat{d}\tau^2$.

$$|M| \text{ var } [\hat{\dot{x}}(t)] = S_6 m + 2(2S_0 m' - 3S_4 m)\tau^2 + 9S_2 m\tau^4$$

and $\quad \partial\text{var } [\hat{\dot{x}}(t)]/\partial t = 0 \quad \text{if} \quad \tau = 0 \quad \text{or} \quad \dot{\tau}_{\pm},$

$$\dot{\tau}^2_{\pm} \equiv (3S_4 m - 2S_0 m')/(9S_2 m)$$

Moreover

$$\frac{|M|}{(3S_4 m - 2S_0 m')} \ \frac{\partial^2 \text{ var } [\hat{x}(t)]}{\partial t^2} = \begin{cases} -4 & \text{at} \quad \tau = 0 \\ +8 & \text{at} \quad \dot{\tau}_{\pm} \end{cases}$$

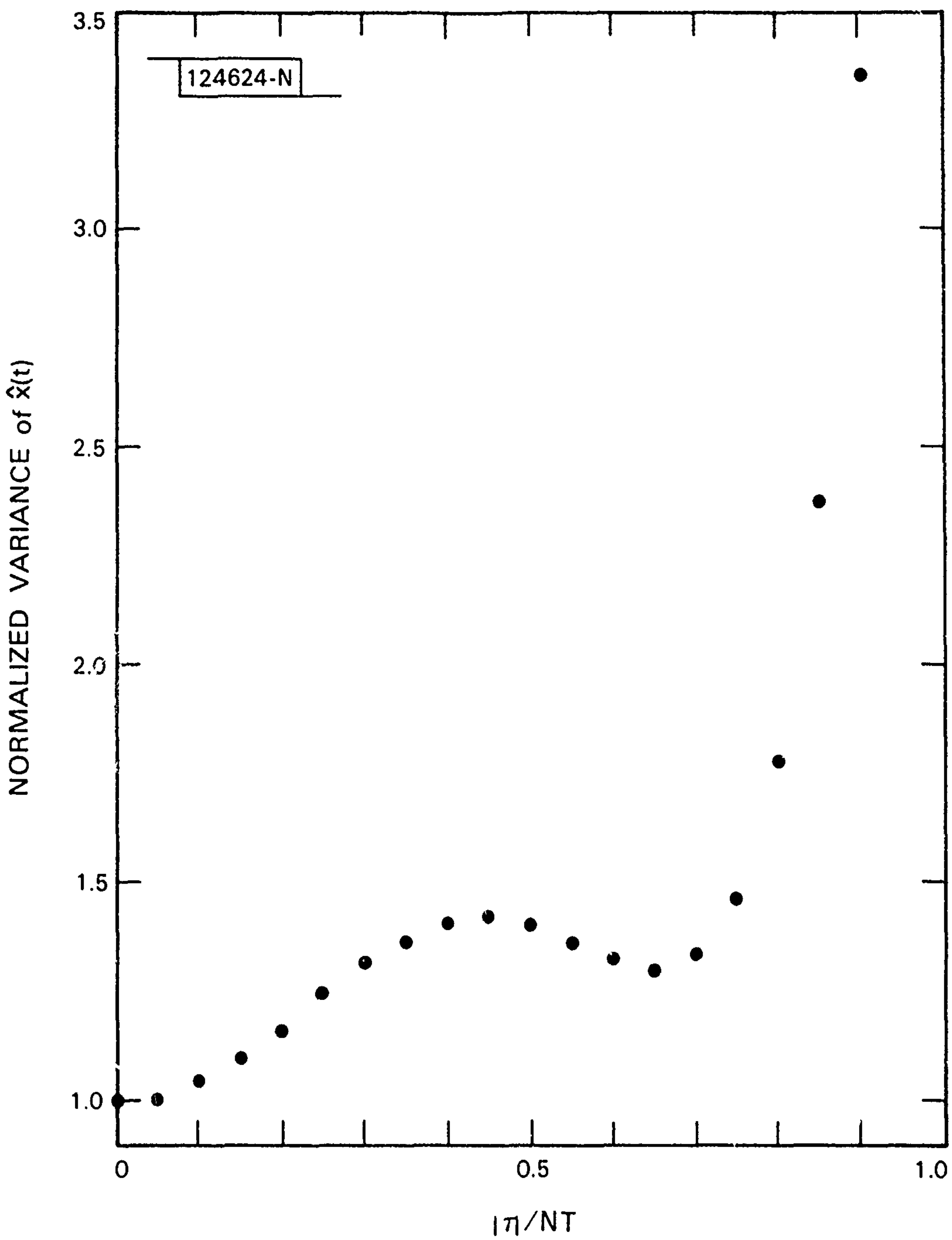

Fig. 2. The normalized variance of the predicted position [var($\hat{x}$(o))=1] in normalized time units τ/NT. Only τ>0 is shown as the var[$\hat{x}(\tau)$] is an even function of τ.

This replicates the type of situation we had in the quadratic model for there are three possibilities:

$3S_4m - 2S_0m'<0$ and neither $\dot{\tau}_\pm$ is real. The absolute minimum for var $[\hat{x}(t)]$ occurs at $\tau = 0$.

$3S_4m - 2S_0m'>0$ and the $\dot{\tau}_\pm$ are the instants when var $[\hat{x}(t)]$ has its absolute minimum. It has a relative maximum at $\tau = 0$.

$3S_4m - 2S_0m'=0$ and $\dot{\tau}_\pm = 0$. The origin is the point of an absolute minimum for var $[\hat{x}(t)]$ as well as a point of inflection.

Before becoming more specific note that

$$|M| \text{ var } [\hat{\dot{x}}(0)] = S_6m$$

$$|M| \text{ var } [\hat{\dot{x}}(\dot{\tau}_\pm)] = S_6m - \frac{(3S_4m - 2S_0m')^2}{9S_2m}$$

Now assume equal spacing of the observations. Then

$$3S_4m - 2S_0m' \to \frac{32}{105} w^3T^8N^{11}(1 + 11/2N) > 0$$

so that var $[\hat{\dot{x}}(t)]$ has a local maximum at $\tau = 0$ and its absolute minima at $\dot{\tau}_\pm$;

$$\dot{\tau}_\pm = \pm \frac{NT}{\sqrt{7}} (1 + 1/2N)$$

Also,

$$|M| \text{ var } [\hat{\dot{x}}(0)] \to \frac{32w^3T^{10}N^{13}(1 + 13/2N)}{315} = \frac{7}{4} |M| \text{ var } [\hat{x}(\dot{\tau}_\pm)]$$

Lastly, we need to examine the variance of $\hat{\ddot{x}}(t) = 2\hat{c} + 6\hat{d}\tau$.

$$|M| \text{ var } [\hat{\ddot{x}}(t)] = 4S_0m' + 36S_2m\tau^2$$

which obviously has a single extremum, a minimum at $\tau = 0$;

$$|M| \text{ var } [\hat{\ddot{x}}(0)] = 4S_0m' + \frac{128}{525} w^3T^8N^{11}(1 + 11/2N).$$

D. Discussion of Alternatives

Among the competitors for night-to-night reductions I have considered are least squares fitting the nightly results to 1) a constant, 2) to a linear form, and 3) to a quadratic form. Then I used the nightly results as 'normal' points to either redo the least squares fits over several nights (for options 1, 2, and 3) or simply average (option 3). With the results just obtained for standard deviations a realistic comparison of the expected variances can be made. Similar analyses based upon linear interpolations of nightly 'normal' points (option 1) or Hermite interpolation (option 2) have been performed. No one of these procedures is better than the overall fit advocated above. Some are noticeably worse.

IV. TESTS

Why does one need an orbital element set for an asteroid today? To recover it at the next opposition ($\sim$ 1.3 years = 1 synodic period). Not to recover it tomorrow, not to recover it at the next dark of the moon. Therefore, it is difficult to get excited about constructing orbital element sets for main-belt minor planets. For an Earth-approaching asteroid one needs an element set for next month. Both 1982HS and 1982SA were discovered by our search program. Unbeknownst to us 1982SA was also discovered prior to us (by two days) by our competition (E. Shoemaker and E. F. Helin). The data discussed below for these objects and 1862 Apollo (recovered by us accidentally) are real observations acquired by us. The data for the Molniya tests are pseudo-observations good to 1".

A. 1982HS and 1982SA

Both of these are inner main-belt, high inclination, high eccentricity minor planets. Since they are minor planets, and orbit the Sun, one needs a heliocentric initial orbit generator, cf. Fig. 1. One also needs to take into account the fact that the observer is on the surface of the Earth. We did this in § IIB wherein the fundamental equations of the problem are given -- the coupled pair of (17) and (20). In practice we solve this as follows: From the observed angular speed we can tell that the minor planet is not close, hence the diurnal parallax correction can be momentarily ignored. We therefore solve the simpler system (21) which resulted in Eq. (22). However, because s = 1 is an exact root of that eighth-order polynomial, we actually use Eq. (22) divided by s - 1, viz.

$$s^7 + s^6 - a(a-2\cos\psi)(s^5 + s^4 + s^3) + a^2 (s^2 + s + 1) = 0$$

This also protects us against unusual geometrical circumstances. Having a guess for $r = sR_S$ we now correct the original observations for diurnal parallax, redo the least squares fit and again solve the above seventh-order equation for s. Note that since our topocentric observations have been adjusted for diurnal parallax this is rigorous. We cycle through this procedure until convergence is achieved on the value of s. Typically this requires at most three iterations.

Our software is set up to perform quadratic, cubic, and quartic fits of both the right ascension and declination separately. The residuals are exhibited in an effort to discern which order is appropriate for which coordinate. For these asteroids fits for ecliptic longitude and ecliptic latitude have also been considered. Obviously no advantage will be gained for Earth-approaching asteroids by such a change of coordinate system.

Tables II and III contain the orbital element sets obtained by us for the four possibilities of quadratic/cubic right ascension/declination fits. Also listed is the closest approximation there is to an initial orbital element set from the Minor Planet Center. Note that we recovered 1982SA a month after its discovery using these orbital element sets (they predict a position within $0^\circ\!.01$ of each other).

B. Apollo

Appollo 1862 is the prototypical Earth-approaching asteroid. We accidentally recovered on April 21, 1982. A total of nine observations were secured that night, three on the next. Table IV shows the results from

TABLE II

ORBITAL ELEMENTS FOR 1982 HS

α,δ fit order:	22	23	32	33	MPC*
a (A.U.)	1.90	1.97	2.19	2.32	2.47
e	0.19	0.21	0.27	0.29	0.33
ω(°)	221.4	222.2	227.5	227.9	229.5
i(°)	18.5	19.8	23.3	25.2	26.4
Ω(°)	44.7	44.4	43.8	43.5	43.0
M_0(°)	326.9	327.4	328.4	330.1	329.5
r (A.U.)	1.62	1.66	1.75	1.80	---

*Differentially corrected, Gauss-Gibbs, distance primed, using more observations (over 6 days) than I.

TABLE III

ORBITAL ELEMENTS FOR 1982 SA

α,δ fit order	22	23	32	33	MPC*
a (A.U.)	1.91	1.88	1.86	1.82	1.85
e	0.14	0.13	0.14	0.14	0.10
$\omega(°)$	61.0	60.9	65.1	65.6	27.8
$i(°)$	21.0	20.1	19.5	18.5	20.0
$\Omega(°)$	350.6	350.3	350.1	349.7	350.1
$M_0(°)$	321.4	321.2	318.5	318.1	346.2
r (A.U.)	1.72	1.69	1.68	1.65	---

*Differentially corrected, Gauss-Gibbs, distance primed, using more observations than I.

TABLE IV

INITIAL CONDITIONS FOR APOLLO

	Observational or Computed	Ephemerides of Minor Planets
r	1.2316 A.U.	1.1779 A.U.
α	$14^h\ 05^m\ 47\overset{s}{.}25$	$14^h\ 05^m\ 49\overset{s}{.}60$
δ	$-13°\ 19'\ 52\overset{''}{.}2$	$-13°\ 19'\ 13\overset{''}{.}4$
R	0.2251 A.U.	0.1711 A.U.
$\dot{r}$		-9.5154×10^{-3} A.U./day
$\dot{\alpha}$	$-265\overset{s}{.}96$/day	$-269\overset{s}{.}772$/day
$\dot{\delta}$	$-763\overset{''}{.}39$/day	$-751\overset{''}{.}29$/day
$\dot{R}$	-1.9220×10^{-2} A.U./day	-9.0496×10^{-3} A.U./day

solving the full coupled set, Eqs. (17, 20) for the topocentric distance. Also shown is the radial velocity result from Eq. (23). In addition I've listed the results of our least squares fits for right ascension, declination and their rates. The second column gives these same quantities as derived by cubic interpolation within the two-day tabulations of the 1982 Ephemerides of Minor Planets. The good agreement is clear for all but $\dot{R}$ - the quantity most sensitive to the angular acceleration.

C. A Rising Molniya

It is very difficult to devise a fair test between the refinement of Laplace's method presented herein and the classical Gauss-Gibbs angles only method. The latter is restrained by three observations and the radius of convergence of the f and g series. Nonetheless, I have restricted the time span of the data set to 80% of the radius of convergence of the f and g series at this place in the orbit and the frequency of data to the GEODSS (not ETS) rate of once every two minutes. Thus a total of eight "observations" spanning 14 minutes were used. Note that only the mean motion is in error (by 13%), everything else is good to 1%. Note too that there has been no mean motion/eccentricity swap. Don't forget too that this is angles only, no distance fixing, over a fourteen minute time span, on the toughest deep space satellite case.

D. A Molniya at Apogee

It shouldn't work for this case and it doens't. An orbit is also superfluous here.

TABLE V

ORBITAL ELEMENTS OF A TYPICAL MOLNIYA

	Calculated	SDC
n	2.2430 rev/day	1.9926 rev/day
e	0.7302	0.7397
ω	313°67	316°13
i	62°86	62°87
Ω	333°46	333°13
M_0	30°45	27°18

V. PARALLAX DETERMINATION

A. Problem Formulation

There is one technique of observation whereby passively acquired angles only measurements can provide distance estimation--when two simultaneous data sets are obtained by separate observers. Since the object (assumed to be an artificial satellite in this Section) is a finite distance away, they will see it projected against different places on the celestial sphere. Knowing this parallactic displacement is equivalent to knowing the distance.

In order to explicitly see that this assertion is true let the artificial satellite's geocentric location be denoted by $\underline{r}$ while that of the observer by $\underline{\rho}$. The satellite's topocentric location $\underline{R}$ is related to these by

$$\underline{r} = \underline{\rho} + \underline{R} \tag{47}$$

In the equatorial coordinate system $\underline{\rho} = \rho\underline{\ell}(\tau,\phi')$ and $\underline{R} = R\underline{\ell}(A,\Delta)$ where τ is the local sidereal time and ϕ' is the observer's geocentric latitude. A and Δ are the topocentric right ascension and declination of the artificial satellite. We know $\underline{\rho}$ and we measure $\underline{\ell}(A,\Delta)$.

Suppose that there are two observers at $\underline{\rho}_1$ and $\underline{\rho}_2$. Suppose further that they simultaneously observe the same satellite. Then

$$\underline{\rho}_1 + \underline{R}_1 = \underline{r} = \underline{\rho}_2 + \underline{R}_2 \tag{48}$$

See Fig. 3 wherein we let $\Delta PO_1O_2 = \theta_1$, $\Delta PO_2O_1 = \theta_2$. We form the scalar products of the topocentric location vectors with the difference of the site vectors,

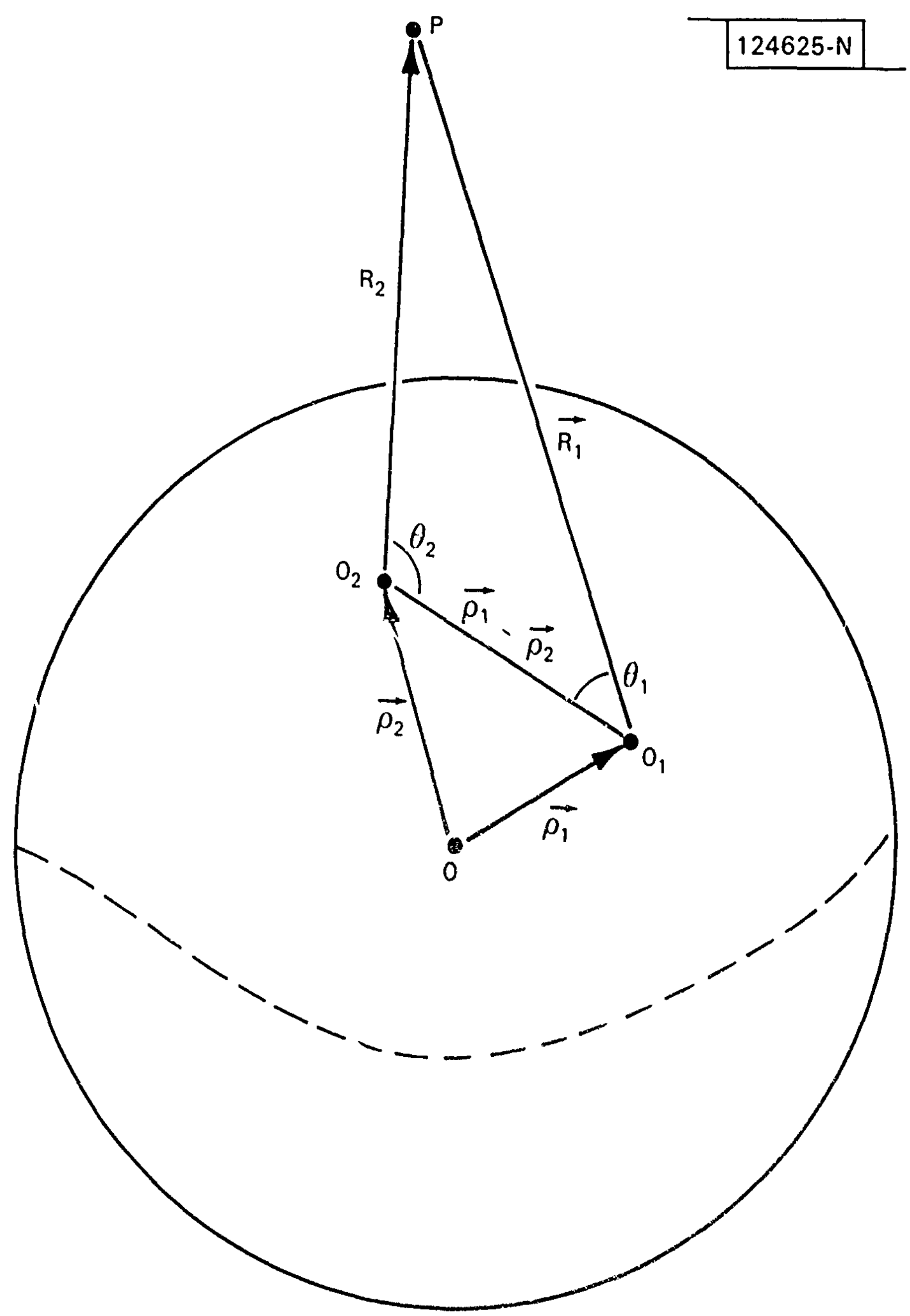

Fig. 3. Geometry for simultaneous observation of an artificial satellite at P by two observers at O_1 and O_2.

$$\underline{R}_2 \cdot (\underline{\rho}_1 - \underline{\rho}_2) = R_2 \, |\underline{\rho}_1 - \underline{\rho}_2| \cos\theta_2$$

$$\underline{R}_1 \cdot (\underline{\rho}_2 - \underline{\rho}_1) = R_1 \, |\underline{\rho}_2 - \underline{\rho}_1| \cos\theta_1 \tag{49}$$

or

$$-\underline{R}_1 \cdot (\underline{\rho}_1 - \underline{\rho}_2) = R_1 \, |\underline{\rho}_1 - \underline{\rho}_2| \cos\theta_1$$

Let $\underline{\rho}_1 - \underline{\rho}_2 = |\underline{\rho}_1 - \underline{\rho}_2| \, \underline{\rho}_{12}$. Then these equations are equivalent to

$$\begin{aligned} \cos\theta_1 &= - \underline{\rho}_{12} \cdot \underline{\ell}(A_1, \Delta_1) \\ \cos\theta_2 &= \underline{\rho}_{12} \cdot \underline{\ell}(A_2, \Delta_2) \end{aligned} \tag{50}$$

and we clearly see that both θ_1 and θ_2 can be calculated. Moreover, using the law of sines in $\Delta O_1 O_2 P$

$$R_1 \csc\theta_2 = R_2 \csc\theta_1 = |\underline{\rho}_1 - \underline{\rho}_2| \csc [\pi - (\theta_1 + \theta_2)] \tag{51}$$

whence

$$\begin{aligned} R_1 &= |\underline{\rho}_1 - \underline{\rho}_2| \sin\theta_2 \csc (\theta_1 + \theta_2) \\ R_2 &= |\underline{\rho}_1 - \underline{\rho}_2| \sin\theta_1 \csc (\theta_1 + \theta_2) \end{aligned} \tag{52}$$

which shows that both R_1 and R_2 may be computed. Finally note that

$$\begin{aligned} |\underline{\rho}_1 - \underline{\rho}_2|^2 &= \rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 \, \underline{\ell}(\tau_1, \phi_1') \cdot \underline{\ell}(\tau_2, \phi_2') \\ &= \rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 [\sin\phi_1' \sin\phi_2' + \cos (\lambda_1 - \lambda_2) \cos\phi_1' \cos\phi_2'] \end{aligned} \tag{53}$$

where λ_1, λ_2 are the observers' longitudes.

B. Analysis of Variance

As demonstrated above one can calculate R_1 and R_2. In an experimental situation the errors associated with such values are of interest too. Since the problem is $1 \longleftrightarrow 2$ symmetrical I arbitrarily choose to investigate the variance of R_1 = var (R_1). First we need expressions of the variances of θ_1 and θ_2.

The expression for var (θ_1) includes three types of terms. One group is the contribution to the variance of θ_1 due to the correlations in the measurement errors of A_1 and Δ_1 with the random errors in the observers' locations. While systematic errors in the site locations will produce biases in A_1 and Δ_1 there should be no coupling between the two sets of random errors. Hence this set of terms is ignored. There is a second group of terms contributing to the variance of θ_1 which is due to random errors in $\underline{\rho}_1$ and $\underline{\rho}_2$ and their cross-correlations. These are not all zero but they should all be small, say $\approx 0\overset{''}{.}01$. As we anticipate that the random measurement errors in A or Δ will be 1-10", this second set will be ignored in comparison. (The full expression for var (θ_1), contains 36 terms - 12 are in the first group and 21 are in the second group).

Given the above (excellent) approximations we have

$$\text{var } (\theta_1) = \left(\frac{\partial\theta_1}{\partial A_1}\right)^2 \text{var } (A_1) + \left(\frac{\partial\theta_1}{\partial \Delta_1}\right)^2 \text{var } (\Delta_1) + 2\left(\frac{\partial\theta_1}{\partial A_1}\right)\left(\frac{\partial\theta_1}{\partial \Delta_1}\right) \text{cov } (A_1, \Delta_1) \tag{54}$$

As there exist observational techniques which ensure that cov $(A_1, \Delta_1) = 0$ we'll use the first two terms to represent the variance of θ_1. Repeating these type of approximations we have

$$\mathrm{var}\,(R_1) = \left(\frac{\partial R_1}{\partial \theta_1}\right)^2 \mathrm{var}\,(\theta_1) + \left(\frac{\partial R_1}{\partial \theta_2}\right)^2 \mathrm{var}\,(\theta_2) \tag{55}$$

since $\mathrm{cov}\,(\theta_1, \theta_2) = 0$. In particular,

$$\mathrm{var}\,(R_1) = |\underline{\rho}_1 - \underline{\rho}_2|^2 \sin\theta^2{}_2 \csc^2(\theta_1 + \theta_2) \Big\{\cot^2(\theta_1 + \theta_2)\,\mathrm{var}\,(\theta_1)$$

$$+ [\cot\theta_1 - \cot\,(\theta_1 + \theta_2)]^2\,\mathrm{var}\,(\theta_2)\Big\} \tag{56}$$

C. Analytical Insight

In order to acquire some feeling for the dependence of the variance of R_1 on $\underline{\rho}_1$, $\underline{\ell}$ (A_1, Δ_1), $\underline{r}$ etc., let us consider a two-dimensional situation. Set $\rho_1 = \rho_2 = \rho$ (for simplicity), $\phi_1{}' = \phi_2{}' = 0$, and $\Delta_1 = \Delta_2 = 0$. Also for simplicity we take $\mathrm{var}\,(A_1) = \mathrm{var}\,(A_2) = \sigma^2$ and call $\lambda_2 - \lambda_1 = \Delta\lambda \geq 0$. Then

$$|\underline{\rho}_1 - \underline{\rho}_2| = 2\rho\sin\,(\Delta\lambda/2)$$

$$2\cos\theta_1 = [\cos(H_1 - \Delta\lambda) - \cos H_1]\csc\,(\Delta\lambda/2)$$

$$2\cos\theta_2 = [\cos(H_2 + \Delta\lambda) - \cos H_2]\csc\,(\Delta\lambda/2) \tag{57}$$

where $H = \iota - A$ is the topocentric hour angle. We find, after performing the indicated differentiations in Eq. (54), that

$$\mathrm{var}\,(\theta_1) = (\sigma/2)^2\,[\sin\,(H_1 - \Delta\lambda) - \sin H_1]^2 \csc^2\theta_1 \csc^2(\Delta\lambda/2)$$

$$\mathrm{var}\,(\theta_2) = (\sigma/2)^2\,[\sin\,(H_2 + \Delta\lambda) - \sin H_2]^2 \csc^2\theta_2 \csc^2(\Delta\lambda/2) \tag{58}$$

Whence,

$$\text{var}(R_1) = \sigma^2\rho^2\sin^2\theta_2 \csc^2(\theta_1+\theta_2)\Big\{\csc^2\theta_1 \cot^2(\theta_1+\theta_2)\cdot$$

$$[\sin(H_1-\Delta\lambda) - \sin H_1]^2 + \csc^2\theta_2[\cot\theta_2 - \cot(\theta_1+\theta_2)]^2\cdot$$

$$[\sin(H_2+\Delta\lambda) - \sin H_2]^2\Big\} \tag{59}$$

Now further specialize to the instance when the satellite is midway between the observers. Then $\theta_1 = \theta_2 = \theta$, $H_1 = -H_2 = H$, and $\theta + H = (\pi + \Delta\lambda)/2$. We find var $(\theta_1) = \sigma^2$ and

$$\text{var}(R) = \frac{\sigma^2\rho^2}{2} \sin^2\theta \sin^2(\Delta\lambda/2)(\csc^4\theta + \sec^4\theta) \tag{60}$$

As $r \to \infty$, $\theta \to \pi/2$ from below. But $R = 2|\underline{\rho}_1 - \underline{\rho}_2| \sec\theta$ now so $\cos\theta = (\rho/R)\sin(\Delta\lambda/2)$. Hence, if $\theta = \pi/2 - \delta\theta$

$$\delta\theta \simeq (\rho/R)\sin(\Delta\lambda/2) \simeq (\rho/r)\sin(\Delta\lambda/2)$$

so

$$\text{var}(R) \to \frac{\sigma^2 r^4}{2\rho^2}\csc^2(\Delta\lambda/2) \tag{61}$$

As an example, suppose $r = 6.61\rho$ (geosynchronous distance), $\sigma = 10''$ (worst case GEODSS), and $\Delta\lambda = 120°$. Then the standard deviation of the topocentric distance is 11km. The important points to note are that var (R) increases as the fourth power of the geocentric distance, the measurement variance, and the square of reciprocal of the site-to-site angular separation.

D. Numerical Results

One can show that when the artificial satellite is equidistant from the observers the minimum of var (R) occurs when all three are coplanar

(i.e. the above case). Hence this simple extension to three dimensions produces no new minima. One suspects that this is true in general and a full numerical treatment of the two dimensional problem suffices to illustrate optimistic distance estimation by parallax techniques. The results are in Table VI, for the standard deviation of R_1, for satellites from the horizon of observer 1 to the equidistant case in steps of 10°. The units of the standard deviation of R_1 are $\sigma\rho$. The multiplier for a 15 km standard deviation with a ten arc second measurement error is 48.5 (eg. an entry in the Table smaller than 48.5 means that the standard deviation of R will be less than 15km if $\sigma = 10''$).

E. Why Is This Subject Here?

This discussion is included in this Report because this is a report about data smoothing. Simultaneous observation is difficult if not impossible. Interspersed observations are much more reasonable. Smoothing the data sets can then be used to produce a pseudo-observation (of higher quality) from each observer at the same epoch. Following that this technique may be employed. Note that the level is well within the range of utility and that the result could be used in an initial orbit determination scheme, differential correction scheme, or as a (severe) constraint on an initial orbit determination method. One anticipates at most one such data point per satellite (due to lighting, visibility, and siting constraints) so that this is only an adjunct to optical methods.

TABLE VI

VARIANCE (R_1) IN UNITS OF $\sigma\rho$

Geocentric Distance = 4.1645ρ

$\Delta\lambda$ =	30°	60°	90°	120°
Altitude				
0°	142.32	32.98	16.02	11.87
10	80.39	24.40	13.80	11.35
20	55.22	19.83	12.66	
30	42.33	17.30	12.23	
40	35.06	16.01		
50	30.90	15.57		
60	28.74			
70	28.04			
Minimum				
70.38	28.04			
51.38		15.57		
33.44			12.21	
16.70				11.28

TABLE VI (continued)

Geocentric Distance = 6.6107ρ

Altitude	Δλ = 30°	60°	90°	120°
0°	402.08	98.42	48.05	33.69
10	234.19	74.22	41.25	31.58
20	165.26	60.95	37.42	30.81
30	129.35	53.27	35.54	
40	108.70	48.99		
50	96.53	47.07		
60	89.84			
70	87.24			
Minimum				
72.38	87.14			
55.03		46.85		
38.17			35.10	
21.93				30.80

TABLE VI (continued)

Geocentric Distance = 13.7509ρ

	Δλ = 30°	60°	90°	120°
Altitude				
0°	1891.6	483.8	238.5	162.0
10	1130.2	370.6	205.1	150.2
20	815.1	307.7	185.4	144.6
30	649.0	270.3	174.6	
40	553.0	248.5	170.5	
50	495.2	237.5		
60	462.3			
70	448.2			
Minimum				
73.84	446.6			
57.78		235.0		
41.89			170.4	
26.26				143.7

REFERENCES

1. L. G. Taff and D. L. Hall, Cel. Mech. 16, 481, 1977.

2. L. G. Taff and D. L. Hall, Cel. Mech. 21, 281, 1980.

3. C. F. Gauss, Theoria Motus, reprinted by Dover Publications, Inc., N.Y., 1963.

4. C. J. Kowal, W. Liller, and B. G. Marsden in "Dynamics of the Solar System", ed. R. L. Duncombe (D. Reidel Publishing Co., Durdrecht, Holland 1979), p. 245.

5. L. G. Taff, "Optical Artificial Satellite Searches", Project Report ETS-44, Lincoln Laboratory, M.I.T. (2 May 1979). DDC-AD-A071248

6. L. G. Taff, "On Gauss' Method of Orbit Determination", Technical Note 1979-49, Lincoln Laboratory, M.I.T. (21 June 1979). DDC-AD-A073776

7. R. W. Miller and C. B. Chang, "Some Analytical Methods for Tracking, Prediction and Interception Performance Evaluations", Project Report RMP-129, Lincoln Laboratory, M.I.T. (9 August 1977).

8. R. W. Miller, "A Lower Bound On Angle-Only Tracking Accuracy", Project Report RMP-149, Lincoln Laboratory, M.I.T. (9 May 1978).

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This report deals with a number of interrelated topics. The common thread is Laplace's method of initial orbit determination based on passively acquired optical data. We discuss this method's principal competitor (that of Gauss), the difficulties of Gauss's technique, and the traditional reasons the Gaussian method is preferred to the Laplacian. We reject this hegemony for a variety of reasons and concentrate on Laplace's method in an era of a surfeit of high quality data. This leads us into a discussion of data smoothing. Once one leaves the raw observatorial data the possibility of combining observations from multiple observers comes to mind and hence the determination of parallax by trigonometrical means. All of this may be applied to two different classes of objects — asteroids and artificial satellites. Our immediate interests are in fast moving asteroids ($>0^\circ\!.5$/day or an abnormally fast ecliptic latitude rate) and high altitude artificial satellites ($P>6^h$). In both instances it is the high inclination and high eccentricity subset which is of especial concern.

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